I: Matrices

(a) If we set $d_2 = I = \begin{bmatrix} 1 \\ 1 \\ ... \\ 1 \end{bmatrix}$ then Regression 3 can be rewritten as:

$$y = \phi d_1 + \mu d_2 + e \tag{1}$$

In Regression 2 we have the condition $d_1 + d_2 = I$ and in Regression 3 we have the condition that $d_2 = I$. These two conditions are independent (i.e. there is no condition that is more restrictive than the other), therefore Regression 2 is no more or no less general than Regression 3.

(b) To simplify the model, consider $d_1 = \begin{bmatrix} a \\ b \end{bmatrix}$. Note that if the i^{th} element of d_1 is 1 then the i^{th} element of d_2 is 0 and if the i^{th} element of d_1 is 0 then the i^{th} element of d_2 is 1. Therefore we have $d_2 = \begin{bmatrix} 1-a \\ 1-b \end{bmatrix}$. Denote $I = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

then we have $I = d_1 + d_2$. In this case, regression 2 cannot be estimated by OLS.

II: Processes

(a) Y_t is covariance stationary. Indeed, we have $E(Y_t) = 0$ and

$$\begin{aligned} \gamma_0 &= var(Y_t) \\ &= var(\epsilon_t + 2.4\epsilon_{t-1} + 0.8\epsilon_{t-2}) \\ &= var(\epsilon_t) + 2.4^2 var(\epsilon_{t-1}) + 0.8^2 var(\epsilon_{t-2}) \text{ (Note that } \epsilon_t \text{ are white noise } E(\epsilon_t\epsilon_j) = 0) \\ &= (1 + 5.76 + 0.64)1 \\ &= 7.4 \end{aligned} \tag{2}$$

$$\begin{aligned} \gamma_1 &= cov(Y_t, Y_{t-1}) \\ &= cov(\epsilon_t + 2.4\epsilon_{t-1} + 0.8\epsilon_{t-3}, \epsilon_{t-1} + 2.4\epsilon_{t-2} + 0.8\epsilon_{t-2}) \\ &= 2.4cov(\epsilon_{t-1}, \epsilon_{t-1}) + 0.8cov(\epsilon_{t-2}, \epsilon_{t-2}) \\ &= 3.2 \end{aligned} \tag{3}$$

$$\begin{aligned} \gamma_2 &= cov(Y_t, Y_{t-2}) \\ &= cov(\epsilon_t + 2.4\epsilon_{t-1} + 0.8\epsilon_{t-2}, \epsilon_{t-2} + 2.4\epsilon_{t-3} + 0.8\epsilon_{t-4}) \\ &= 0.8cov(\epsilon_{t-2}, \epsilon_{t-2}) \\ &= 0.8 \end{aligned} \tag{4}$$

and

$$\gamma_j = 0 \text{ for } j > 2 \tag{5}$$

(b) We can rewrite the $AR(2)$ process as:

$$(1 - 0.9L)(1 - 0.2L)Y_t = \epsilon_t \tag{6}$$

or

$$Y_t = \frac{\epsilon_t}{(1 - 0.9L)(1 - 0.2L)} \tag{7}$$

We have:

$$(1 - 0.9L)^{-1} = \sum_{j=0}^{\infty} (0.9)^j L^j \tag{8}$$

and

$$(1 - 0.2L)^{-1} = \sum_{j=0}^{\infty} (0.2)^j L^j \tag{9}$$

Therefore

$$\begin{aligned} [(1 - 0.9L) * (1 - 0.2L)]^{-1} &= \sum_{i=0}^{\infty} (0.9)^i L^i \sum_{j=0}^{\infty} (0.2)^j L^j \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (0.9)^i (0.2)^j L^{i+j} \end{aligned} \tag{10}$$

Hence we can invert the $AR(2)$ process into an $MA(\infty)$ process

$$Y_t = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (0.9)^i (0.2)^j \epsilon_{t-i-j} \tag{11}$$

We then have:

$$E(Y_t) = 0 \tag{12}$$

$$\gamma_0 = cov(\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (0.9)^i (0.2)^j \epsilon_{t-i-j}, \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} (0.9)^k (0.2)^l \epsilon_{t-k-l}) \tag{13}$$

Note that $cov(0.9)^i (0.2)^j \epsilon_{t-i-j}, (0.9)^k (0.2)^l \epsilon_{t-k-l} \neq 0$ when $i + j = k + l$ or

$$l = i + j - k \tag{14}$$

. Then we have:

$$\begin{aligned}\gamma_0 &= cov(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(0.9)^i(0.2)^j\epsilon_{t-i-j}, \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(0.9)^k(0.2)^l\epsilon_{t-k-l}) \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(0.9)^{i+k}(0.2)^{i+2j-k}\end{aligned} \tag{15}$$

$$\gamma_1 = cov(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(0.9)^i(0.2)^j\epsilon_{t-i-j}, \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(0.9)^k(0.2)^l\epsilon_{t-1-k-l}) \tag{16}$$

Note that $cov(\epsilon_{t-i-j}, \epsilon_{t-1-k-l}) \neq 0$ if $i + j = k + l + 1$ or:

$$l = i + j - k - 1 \tag{17}$$

when $k < i + j$. So,

$$\begin{aligned}\gamma_1 &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{i+j-1}(0.9)^i(0.2)^j(0.9)^k(0.2)^{i+j-k-1} \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{i+j-1}(0.9)^{i+k}(0.2)^{i+2j-k-1}\end{aligned} \tag{18}$$

Similarly,

$$\gamma_2 = cov(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(0.9)^i(0.2)^j\epsilon_{t-i-j}, \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(0.9)^k(0.2)^l\epsilon_{t-2-k-l}) \tag{19}$$

We can apply the same token by noting that $cov(\epsilon_{t-i-j}, \epsilon_{t-2-k-l}) \neq 0$ if $i + j = k + l + 2$