Chapter 4: Autoregressive processes - ARMA

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First order Autoregressive (AR) process

- Let $\{\epsilon_t\}$ be white noise.
- The first order AR(1) process has the following form:

$$Y_t = c + \phi Y_{t-1} + \epsilon_t \tag{1}$$

- We assume that $|\phi| < 1$. Why?
- We also assume that AR(1) is covariance stationary and ergodic for the mean.
- What is the expected value of Y_t?
- What is the variance of Y_t?

Impulse response

- What is the auto-covariance of Y_t?
- What is the autocorrelation of Y_t?
- What is effect of a one-unit increase in ϵ_t on Y_{t+j}?

The pth order Autoregressive (AR) process

- Let $\{\epsilon_t\}$ be white noise.
- The pth order AR(p) process has the following form:

$$Y_t = c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + ... + \phi_p Y_{t-p} + \epsilon_t \qquad (2)$$

- We again assume that $|\phi| < 1$, and that AR(p) is covariance stationary and ergodic for the mean.
- We can rewrite AR(p) as

$$Y_t = c + \phi_1 L Y_t + \phi_2 L^2 Y_t + ... + \phi_p L^p Y_t + \epsilon_t \qquad (3)$$

Properties of the AR(p) process

- What is the expected value of Y_t?
- What is the variance of Y_t?
- What is the auto-covariance of Y_t?
- What is the autocorrelation of Y_t?

ARMA(p,q) process

- Let $\{\epsilon_t\}$ be white noise.
- The ARMA(p,q) includes both autoregressive and moving average terms:

$$Y_t = c + \phi_1 Y_{t-1} + \ldots + \phi_p Y_{t-p} + \epsilon_t + \theta_1 \epsilon_{t-1} + \ldots + \theta_q \epsilon_{t-q} \quad (4)$$

- In lag operator form:

$$(1 - \phi_1 L - \ldots - \phi_p L^p) Y_t = c + (1 + \theta_1 L + \ldots + \theta_q L^q) \epsilon_t \quad (5)$$

Example: ARMA(1,2) process

- Let $\{\epsilon_t\}$ be white noise.
- The ARMA(1,2) includes both autoregressive and moving average terms:

$$Y_t = c + \phi_1 Y_{t-1} + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2}$$

- In lag operator form:

$$(1 - \phi_1 L) Y_t = c + (1 + \theta_1 L + \theta_2 L^2) \epsilon_t$$

Definition

- Let consider the covariance stationary process $\{Y_t\}$.
- Denote γ_j the jth autocovariance of this process.
- The *autocovanriance generating function* is defined as

$$g_Y(z) = \sum_{j=-\infty}^{\infty} \gamma_j z^j \tag{6}$$

Complex numbers

- The *imaginary unit* is defined as the square root of -1.

$$i = \sqrt{-1} \tag{7}$$

- Denote x and y two real numbers. The complex numbers have the following form:

$$z = x + iy \tag{8}$$

Properties of Complex numbers

- The *complex conjugate* is defined as

$$\bar{z} = x - iy \tag{9}$$

- The absolute square

$$|z|^2 = z^2 = z\bar{z} = x^2 + y^2 \tag{10}$$

- z can be written in "phasor" form

$$z = |z|(cos(\theta) + i * sin(\theta)) = e^{-i\theta} \tag{11}$$

Population spectrum

- Let z be a complex number on the complex unit circle:

$$z = cos(w) + isin(w) = e^{-iw} \tag{12}$$

- The *population spectrum* of Y is the autocovariance generating function evaluated at z divided by 2π.

$$s_Y(w) = \frac{1}{2\pi} g_Y(z = e^{-iw}) = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \gamma_j z^j \tag{13}$$

Example 1: MA(1)

- Recall the MA(1) process

$$Y_t = \mu + \epsilon_t + \theta\epsilon_{t-1}$$

- Its autocovariance are

$$\gamma_0 = (1+\theta^2)\sigma^2$$
$$\gamma_1 = \theta\sigma^2$$
$$\gamma_j = 0 \text{ if } j > 1$$

- Its autocovariance generating process is

$$\begin{aligned} g_Y(z) &= \sum_{j=-\infty}^{\infty} \gamma_j z^j \\ &= (\theta\sigma^2)z^{-1} + (1+\theta^2)\sigma^2 + (\theta\sigma^2)z \\ &= \sigma^2(\theta z^{-1} + (1+\theta^2) + \theta z) \\ &= \sigma^2(1+\theta z)(1+\theta z^{-1}) \end{aligned}$$

Example 2: MA(q)

- Recall the MA(q) process:

$$\begin{aligned} Y_t &= \mu + \theta_0\epsilon_t + ... + \theta_q\epsilon_{t-q} \\ &= \mu + \theta(L)\epsilon_t \end{aligned}$$

  with $\theta(L) = \theta_0 + \theta_1 L + ... + \theta_q L^q$

- Show that the autocovariance generating function of MA(q) is

$$\begin{aligned} g_Y(z) &= \sigma^2(\theta_0 + \theta_1 z + ... + \theta_q z^q)(\theta_0 + \theta_1 z^{-1} + ... + \theta_q z^{-q}) \\ &= \sigma^2\theta(z)\theta(z^{-1}) \end{aligned}$$

Example 3: AR(1)

- Recall the AR(1) process:

$$Y_t = \mu + \phi Y_{t-1} + \epsilon_t$$

- We can rewrite this process as

$$Y_t - \mu = (1 - \phi L)\epsilon_t$$

- Show that the autocovariance generating process is

$$g_Y(z) = \frac{\sigma^2}{(1 - \phi z)(1 - \phi z^{-1})}$$

Example 4: ARMA(p,q)

- Recall the ARMA(p,q) process:

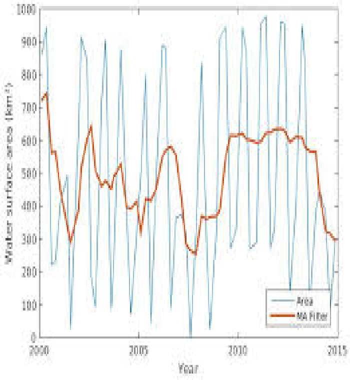
$$Y_t = c + \phi_1 Y_{t-1} + ... + \phi_p Y_{t-p} + \epsilon_t + \theta_1 \epsilon_{t-1} + ... + \theta_q \epsilon_{t-q}$$

- Show that the autocovariance generating process is

$$g_Y(z) = \frac{\sigma^2(1 + \theta_1 z + ... + \theta_q z^q)(1 + \theta_1 z^{-1} + ... + \theta_q z^{-q})}{(1 - \phi z - ...\phi_p z^p)(1 - \phi z^{-1} - ... - \phi_p z^{-p})}$$

Filtered data

- Sometimes we need to filter the data in a particular way before analyzing it.
- For instance, we might need to subtract the seasonal element in the process: sales in December and January are normally stronger than in the other months.



- This is call filtering

Example: filtered MA(1)

- Suppose we have an MA(1) process

$$\begin{aligned} Y_t &= \epsilon_t + \theta \epsilon_{t-1} \\ &= (1 + \theta L)\epsilon_t \end{aligned}$$

- The autocovariance generating process is

$$g_Y(z) = \sigma^2 (1 + \theta z)(1 + \theta z^{-1})$$

- We generate another process by analyzing the change in Y_t over its value the previous period

$$\begin{aligned} X_t &= Y_t - Y_{t-1} \\ &= (1 - L)Y_t \\ &= (1 - L)(1 + \theta L)\epsilon_t \\ &= (1 + (\theta - 1)L - \theta L^2)\epsilon_t \end{aligned}$$

Autocovariance generating function of a filtered process

- By filtering the data, we generate a MA(2) process.
- What is its autocovariance genearating process ?
- In general, if

$$X_t = h(L) Y_t$$

- then

$$g_X(z) = h(z)h(z^{-1})g_Y(z) \tag{14}$$

MA(1) process

- Consider the MA(1) process

$$Y_t - \mu = (1 + \theta L)\epsilon_t \tag{15}$$

  with $|\theta| < 1$.
- Note that

$$\begin{aligned}(1 + L)^{-1} &= (1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + ...) \\ &= \sum_{j=0}^{\infty} (-1)^j \theta^j L^j\end{aligned}$$

- We can invert the MA(1) process by multiplying both sides of Equation 15 with $(1 + L)^{-1}$

$$(1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + ...)(Y_t - \mu) = \epsilon_t \tag{16}$$

- We then have an $AR(\infty)$ process

Invertible process

- Note that since $|\theta| < 1$, $\lim_{j\to\infty} \theta^j = 0$. In other words, $\sum_{j=0}^{\infty}(-1)^j\theta^j L^j$ is well defined.
- In this case we say that the $MA(1)$ process with $|\theta| < 1$ is invertible.
- In contrast, if $|\theta| > 1$ we have a non-invertible process as $\sum_{j=0}^{\infty}(-1)^j\theta^j L^j$ explodes.

Two sides of the same coin

- Recall that the autocovariance generating process of the invertible MA(1) process is

$$g_Y(z) = \sigma^2(1 + \theta z)(1 + \theta z^{-1})$$

- Now consider the non-invertible MA(1) process

$$\tilde{Y}_t - \mu = (1 + \tilde{\theta} L)\tilde{\epsilon}_t \tag{17}$$

with $\tilde{\theta} = 1/\theta$.

- The autocovariance generating process of this non-invertible MA(1) process is

$$\begin{aligned} g_{\tilde{Y}}(z) &= \tilde{\sigma}^2(1 + \tilde{\theta} z)(1 + \tilde{\theta} z^{-1}) \\ &= \tilde{\sigma}^2(1/(\tilde{\theta} z) + 1)(\tilde{\theta} z)(1/(\tilde{\theta} z^{-1}) + 1)(\tilde{\theta} z^{-1}) \\ &= \tilde{\sigma}^2\tilde{\theta}^2(1 + \theta z^{-1})(1 + \theta z) \end{aligned}$$

Two sides of the same coin (cont')

- We can see that Y_t and $\tilde{Y}_t$ have the same autocovariance generating function.
- In other words, any MA(1) process can be represented as invertible or non-invertible.

Fundamental innovation

- Recall that we can invert the MA(1) process according to Equation 16

$$\begin{aligned}\epsilon_t &= (1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \ldots)(Y_t - \mu) \\ &= (Y_t - \mu) - \theta(Y_{t-1} - \mu) + \theta^2(Y_{t-2} - \mu) - \ldots\end{aligned}$$

- The white noise ϵ_t is called the *fundamental innovation*, which can be calculated from the *past* values of the data Y_t.