Chapter 3: Stationary processes - MA processes

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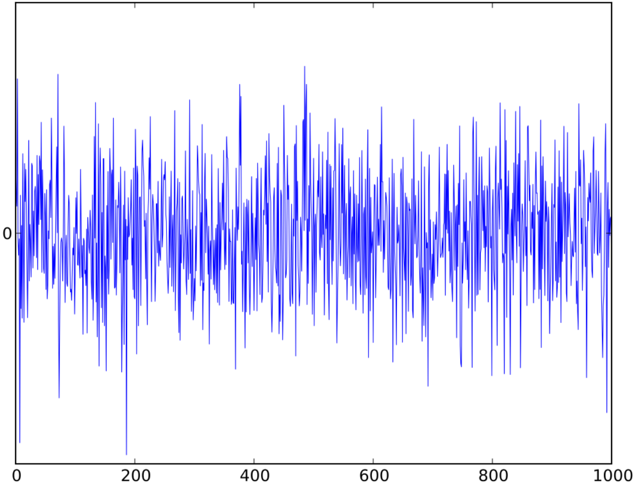
White noise

- Consider a collection of T independent and identically distributed (i.i.d.) variables ϵ:

$$\{\epsilon_1, \epsilon_2, ..., \epsilon_T\}$$

- We assume that $\epsilon_t \sim N(0, \sigma^2)$ for any t.
- This is called the Gaussian white noise process.

White noise process



Expectations of a process

- Suppose this white noise process generates a sequence:

$$Y_t = \mu_t + \epsilon_t$$

- this sequence can be observed as:

$$\{y_1, y_2, ..., y_T\}$$

- This sequence $\{Y_1, Y_2, ..., Y_T\}$ are random variables with expectations (unconditional):

$$E(Y_t) = \mu_t \tag{1}$$

Vector of observations

- Given a particular realization $\{y_t^{(1)}\}_{t=-\infty}^{\infty}$, construct a vector X_t that consists of $[j+1]$ most recent observations on y:

$$\mathbf{X}_t^{(1)} = \begin{bmatrix} y_t^{(1)} \\ y_{t-1}^{(1)} \\ \vdots \\ y_{t-j}^{(1)} \end{bmatrix} \tag{2}$$

Vector of observations(cont')

- If we have another realization $\{y_t^{(2)}\}_{t=-\infty}^{\infty}$, construct a vector X_t that consists of $[j+1]$ most recent observations on y:

$$\mathbf{X}_t^{(2)} = \begin{bmatrix} y_t^{(2)} \\ y_{t-1}^{(2)} \\ \vdots \\ y_{t-j}^{(2)} \end{bmatrix} \tag{3}$$

Distribution

- We can think of this vector $\mathbf{X}_t$ is a random variable that depends the realization of the sequence Y
- It has an expectation

$$E(\mathbf{X}_t) = \begin{bmatrix} \mu_t \\ \mu_{t-1} \\ \vdots \\ \mu_{t-j} \end{bmatrix} \tag{4}$$

Autocovariance

- Its variance matrix is given by:

$$Var(\mathbf{X}_t) = \begin{bmatrix} \sigma^2 & \gamma_{t,t-1} & \cdots & \gamma_{t,t-j} \\ \gamma_{t-1,t} & \sigma^2 & \cdots & \gamma_{t-1,t-j} \\ & & \vdots & \\ \gamma_{t-j,t} & \gamma_{t-j,t-1} & \cdots & \sigma^2 \end{bmatrix} \tag{5}$$

- where $\gamma_{t,t-j} = E(Y_t - \mu_t)(Y_{t-j} - \mu_{t-j})$
- this is the covariance of Y_t with its own lagged value: hence autocovariance.

Weak Stationarity

- A weakly stationary (or covariance stationary) process is when the mean and autocovariance are time-invariant:

$$E(Y_t) = \mu \text{for all t}$$
$$E(Y_t - \mu)(Y_{t-j} - \mu) = \gamma_j \text{for all t and j}$$

- If $\mu_t = \mu$ for any t then the process 5 is weakly stationary:

$$E(Y_t) = \mu$$
$$E(Y_t - \mu)(Y_{t-j} - \mu) = 0 \text{ if j} \neq 0$$
$$E(Y_t - \mu)(Y_t - \mu) = \sigma^2$$

Strict Stationarity

- A process is strict stationary if for any values of $j_1, j_2, ..., j_n$ the joint distribution of $(Y_t, Y_{t+j1}, ..., Y_{t+jn})$ depends only on the intervals that separate the dates $j_1, j_2, ..., j_n$ and not the date itself t.
- If the second moments are finite then a strictly stationary process is also weakly stationary (or covariance-stationary).
- Is weakly stationary process also strictly stationary?

Gaussian process

- A process is Gaussian if the joint distribution of $(Y_t, Y_{t+j1}, ..., Y_{t+jn})$ is Gaussian for any values of $j_1, j_2, ..., j_n$.
- A Gaussian process is strictly stationary.

Ergodicity

- Given a realization $\{y_1^{(1)}, y_2^{(1)}, ..., y_T^{(1)}\}$ we define a time average as:

$$\bar{y} := (1/T) \sum_{t=1}^{T} y_t^{(1)} \tag{6}$$

- A weakly stationary process is ergodic for the mean if its time average *converges in probability* to $E(Y_t)$ as $T \rightarrow \infty$.
- A covariance stationary process is ergodic for the mean if

$$\sum_{j=0}^{\infty} |\gamma_j| < \infty \tag{7}$$

White noise

- The building blocks for all the processes is the white noise.
- By definition, the white noise process has zero mean, variance σ^2 and zero auto-covariance.
- Recall the Gaussian white noise?

First order Moving Average (MA) process

- Let $\{\epsilon_t\}$ be white noise.
- The first order MA(1) process has the following form:

$$Y_t = \mu + \epsilon_t + \theta\epsilon_{t-1} \quad (8)$$

- What is the expected value of Y_t?
- What is the variance of Y_t?
- What is the auto-covariance of Y_t?
- Is MA(1) covariance-stationary?
- Is MA(1) ergodic for the mean?

Autocorrelation

- The *jth* autocorrelation of a covariance stationary process is defined as
$$\rho_j := \gamma_j/\gamma_0 \tag{9}$$
- If we expand this formula we have
$$\rho_j = \frac{Cov(Y_t, Y_{t-j})}{\sqrt{Var(Y_t)}\sqrt{Var(Y_{t-j})}} = Corr(Y_t, Y_{t-j})$$
- Why $\gamma_0 = \sqrt{Var(Y_t)}\sqrt{Var(Y_{t-j})}$?

Properties of MA(1)

- What is the value of the first autocorrelation ρ_1?

- What are the values of higher autocorrelations?

The qth order Moving Average (MA) process

- Let $\{\epsilon_t\}$ be white noise.
- The qth order MA(q) process has the following form:

$$Y_t = \mu + \epsilon_t + \theta_1 \epsilon_{t-1} + ... + \theta_q \epsilon_{t-q} \tag{10}$$

- What is the expected value of Y_t?
- What is the variance of Y_t?
- What is the auto-covariance of Y_t?
- Is MA(q) covariance-stationary?
- Is MA(q) ergodic for the mean?

The infinite order Moving Average (MA) process

- Let $\{\epsilon_t\}$ be white noise.
- The qth order MA(q) process has the following form:

$$Y_t = \mu + \sum_{j=0}^{\infty} \theta_j \epsilon_{t-j} \tag{11}$$

- We assume that $\sum_{j=0}^{\infty} |\theta_j| < \infty$
- What is the expected value of Y_t?
- What is the variance of Y_t?
- What is the auto-covariance of Y_t?
- Is MA(q) covariance-stationary?
- Is MA(q) ergodic for the mean?