Chapter 2: Difference Equations - Lag Operators

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Difference equation

- A difference equation is an expression relating a variable y to its previous values.
- The dynamic equation:

$$y_t = \phi y_{t-1} + w_t \tag{1}$$

- Example: y_t is asset price and w_t is the interest rate.
- Question: what are the effects of changes in the value of w on y?

Recursive substitution

- If we know the starting value of y (e.g y_0) and all the values of w_t we can compute the values of y_t.
- In particular

$$y_2 = \phi y_1 + w_2 = \phi(\phi y_0 + w_1) + w_2 = \phi^2 y_0 + \phi w_1 + w_2$$

- and

$$y_3 = \phi y_2 + w_3 = \phi^3 y_0 + \phi^2 w_1 + \phi w_2 + w_3$$

- more generally

$$y_t = \phi^t y_0 + \phi^{t-1} w_1 + ... + \phi w_{t-1} + w_t \tag{2}$$

The effects of w on y

- What are the effects of w_s on y_t for $s = 1, ..., t$?
- From Equation 2 we have:

$$\frac{\partial y_t}{\partial w_s} = \phi^{t-s}$$

- if we increase w_s by one unit then y_t will increase by ϕ^{t-s} units.
- Note that the effect only depends on the gap between t and s. In other words, the effect of interest rate in 2000 on the asset price in 2005 is the same as the effect of interest rate in 2010 on the asset price in 2015.

Discussion

- If $-1 < \phi < 1$, the effects of w_s on y_t decays geometrically toward zero.
$$\lim_{t->\infty} \frac{\partial y_t}{\partial w_s} = 0$$
- the interest rates in 1000 has almost no effects on the asset price in 2000 if $-1 < \phi < 1$.
- In contrast, if $1 < \phi$ or $\phi < -1$, the effects of w_s on y_t increase exponentially.
$$\lim_{t->\infty} \frac{\partial y_t}{\partial w_s} = \infty$$
- To ensure a stable system, we require $|\phi| < 1$.

Temporary/permanent change

- When we change the value of ONLY one w (say w_1), we call that a **temporary** change.
- In the stable system, the effect of this temporary change will die out eventually.
- In contrast, if we change ALL the w, we call that a **permanent** change (the long run effect).
- In this case, the long run effect on y_t will be

$$\frac{\partial y_t}{\partial w_1} + \frac{\partial y_t}{\partial w_2} + ... + \frac{\partial y_t}{\partial w_t} = \phi^{t-1} + \phi^{t-2} + ... + 1 = \frac{1-\phi^t}{1-\phi}$$

Definition

- We can define a lag operator L as:

$$Ly_t = y_{t-1} \tag{3}$$

- We then can rewrite the first-order Difference equation 1 as follows:

$$y_t = \phi L y_t + w_t \tag{4}$$

Mathematical manipulations

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- Multiply both sides by the following operator $(1 + \phi L + \phi^2 L^2 + ... + \phi^{t-1} L^{t-1})$ we have:

$$(1 - \phi^t L^t) y_t = (1 + \phi L + \phi^2 L^2 + ... + \phi^{t-1} L^{t-1}) w_t$$

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- which brings:

$$y_t - \phi^t y_0 = w_t + \phi w_{t-1} + \phi^2 w_{t-2} + ... + \phi^{t-1} w_1$$